Int. J. Solids Structures Vol. 30, No. 11, pp. 1513–1526, 1993
Printed in Great Britain

0020-7683/93 $6.00+ .00

THE HERTZ FRICTIONAL CONTACT BETWEEN NONLINEAR ELASTIC ANISOTROPIC BODIES (THE SIMILARITY APPROACH)

F. M. BORODICH
Division of Technical Mechanics, Moscow Institute of Radio Engineering, Electronics and Automation, 78 Vernadsky prosp., Moscow, 117454, Russia

(*Received* 24 *October* 1991; *in revised form* 20 *August* 1992)

Abstract—Self-similarity and similarity aspects of the three-dimensional Hertz problem of contact between two nonlinear elastic anisotropic bodies are considered under various boundary conditions: frictionless, adhesive, frictional. It is assumed that in a problem with friction the contact region consists of the following parts: in the inner part the interfacial friction must be sufficient to prevent any slip taking place between the bodies and in the outer part the friction must satisfy the Coulomb frictional law. The qualitative conclusions on the character of changes to the contact region and approach of the bodies are described exactly. A formula similar to the empirical law of Meyer is drawn up exactly without solving the field equations. The results are applied in an analysis of the Hertz impact problem with friction.

1. INTRODUCTION

The complete analytic solutions to the Hertz problems of finding stress fields that arise when two deformable bodies are pressed together were obtained only for isotropic or transversely isotropic linear elastic bodies. In 1881, Hertz analysed the three-dimensional problem of normal frictionless contact between two homogeneous, isotropic, linear elastic bodies making the following approximating assumptions: (i) the shapes of the bodies are described by forms; (ii) the size of the contact region is small with respect to the smallest radius of curvature of the two bodies and the boundary-value problems for both bodies are formulated as for half-spaces; (iii) the contact region is an ellipse. Working with these assumptions Hertz applied some known results of potential theory.

The solutions of axisymmetric problems of frictionless contact, in the cases when the shapes of the punches are described by a monomial, were obtained by Shtaerman (1939) (the degree of the monomial is even) and Galin (1953) (the degree of the monomial is positive rational). Kil'chevskii (1976) applied the Shtaerman results (1939) to the consideration of axisymmetric impact problems.

Some axisymmetric adhesive contact problems for linear elastic isotropic bodies were solved by Mossakovskii (1954, 1963), Goodman (1962) and Spence (1968). Spence showed that, for the indentor described by a power law, the solution of this problem is self-similar. He also solved the same problem for a punch with the shape given by a general polynomial.

The frictional self-similar contact problems in the axisymmetrical case were considered by Spence (1975). Spence's results were extended to transversely isotropic bodies by Turner (1980). The various contact problems of isotropic elasticity with friction were also considered by Galin (1953), Vermeulen and Johnson (1964), Keer (1967), Gladwell (1980), Spektor (1981), Bryant and Keer (1982), Johnson (1985), Kalker (1985), Hills and Sackfield (1987), etc.

The three-dimensional problem of frictionless contact and impact of anisotropic bodies (in Hertz's approximation) was analysed by Willis (1966). In this analysis the functional form of the pressure distribution between the bodies was found explicitly but a complete solution was not obtained. It is well known that even the displacements produced by a concentrated normal load on an anisotropic half-space cannot be found analytically; correspondingly, it is unreasonable to expect the complete analytic solution to the anisotropic contact problem. For nonlinear bodies the Hertzian contact problems were also only solved approximately [see references in Johnson (1985)].

Evidently we cannot find the analytical solution to the considered problem of the frictional contact between nonlinear elastic bodies but we extract as much insight into the behavior of the solution as possible, regarding contact regions and character of dependence on load without actually solving any field equations. Our consideration is based on similarity methods which have been used only recently to consideration of problems of contact between nonlinear axisymmetric and three-dimensional anisotropic bodies in frictionless cases only [see e.g. Borodich (1988b, 1989, 1990), Hill *et al.* (1989) and Storåkers (1989)].

It is shown that if the distance between two bodies is determined by an arbitrary positive and homogeneous function of plane Cartesian coordinates and the stress potential is a homogeneous function of the strain tensor, the solution of such a problem is self-similar for any of the following boundary conditions: adhesive, frictional or frictionless. It is shown that the similarity consideration provides the functional forms of the tangential displacements within the inner contact region. We also consider a two-parameter similarity transformation which transforms the solution of one contact problem into the solution of another. Using the similarity properties of the solutions we obtain the qualitative conclusions on the character of changes of the contact region and approach of the bodies. These conclusions generalize the empirical Meyer's hardness law and semi-empirical formulae of Bowden and Tabor (1964). We apply these conclusions to the analysis of some known experimental results and to the problem of Hertzian collision between two nonlinear elastic bodies. In the cases of isotropic or transversely isotropic linear elastic bodies the impact problems are solved exactly using the results of Spence (1968) and Turner (1980).

2. FORMULATION OF THE CONTACT PROBLEM

We consider two bodies contacting together so that the resultant force between them is P and there is only contact over a small region of the surface of each. The Hertz contact problem is formulated in detail in many works [see e.g. Gladwell (1980), Johnson (1985) and Willis (1966)]. Hertz assumed that the distance between two bodies is determined by a quadratic form. Here we consider the bodies with more general shapes, the particular case of which are quadratic forms.

Let us place the origin of Cartesian x_1, x_2, x_3^+- and x_1, x_2, x_3^--coordinates at the point of initial (in the unstressed state) contact between two bodies. We combine the Ox_1x_2 plane with the common plane tangent to the surfaces of the bodies at the point of the contact and the x_3^+ and x_3^--axes are directed along the inward normals of the bodies.

We shall denote the quantities referring to the body $x_3^+ \geqslant 0$ by a superscript "plus", and those referring to the second body by a superscript "minus" sign. Then the equations of the surfaces of the bodies are given as

$$x_3^+ = -f^+(x_1,x_2), \quad x_3^- = -f^-(x_1,x_2), \tag{1}$$

where f^+ and f^- are certain functions of the coordinates.

In Hertzian formulation the resultants of compressive forces are always assumed to lie on the x_3^+ and x_3^--axes.

After the bodies are compressed together, displacements u^+ and u^- are generated. Suppose that the relative approach of the centers of gravity is $\alpha > 0$. Then, at all points of the region of contact we have

$$u_3^+ + u_3^- = \alpha - f, \quad f \equiv f^+ + f^-. \tag{2}$$

In Hertzian formulation it is also supposed that the solution valid near the contact region could be found by replacing each body by a half-space, while retaining the boundary condition (2). Then the half-space $x_3^+ \geqslant 0$ has a right-hand Cartesian system, but $x_3^- \geqslant 0$ has a left-hand system.

Finally, when the forms of the bodies $f^+(x_1, x_2)$ and $f^-(x_1, x_2)$ and the compressing force P are given, we must find the region G on the boundary plane $\mathbf{R}^2$ of the half-spaces at the points at which the bodies are in mutual contact, the quantity α represents the elastic approach of the bodies, displacements $\mathbf{u}^+$ and $\mathbf{u}^-$ and stresses σ_{ij}^+ and σ_{ij}^- the points of half-spaces $(\mathbf{R}_+^3)^+$ and $(\mathbf{R}_+^3)^-$.

In the problems considered, the compressing force P is included as a parameter, so we shall write it among the arguments of the unknown quantities. Note that we may take as parameters of the contact problem some other quantities, e.g. the approach of bodies α or the size of the contact region l. The parameter l was used by Mossakovskii (1963), Spence (1968) and Hill and Storåkers (1990), the parameter α was used by Galanov (1981).

It is convenient to introduce the conventions that if quantities are written without superscripts plus and minus then they are applied equally to both bodies and that Latin suffixes take the values 1, 2, 3.

The sought quantities must satisfy in each of the bodies the following conditions:

The equations of equilibrium

$$\sigma_{ij,j}(\mathbf{x}, P) = 0, \tag{3}$$

where differentiation with respect to x_j is denoted by $,j$ and the summation convention is employed.

The constitutive relationships for elastic bodies one can write in the following form:

$$\sigma_{ij} = \partial U(\varepsilon_{kl})/\partial \varepsilon_{ij}, \tag{4}$$

where U is the stress potential (elastic energy). The material behavior characterized by (4) may be anisotropic or isotropic, depending on the form of the stress potential U.

The conditions at infinity:

$$\mathbf{u}(\mathbf{x}, P) \to 0, \quad (x_1^2 + x_2^2 + x_3^2) \to \infty. \tag{5}$$

The boundary conditions on the $x_3 = 0$ surface which include the following:

(i) conditions in the contact region, i.e. for $(x_1, x_2) \in \mathbf{R}^2 \backslash G(P)$

$$\sigma_{3j}(x_1, x_2, O, P) = 0; \tag{6}$$

(ii) integral conditions

$$\iint_{G(P)} \sigma_{33}(x_1, x_2, O, P)\, \mathrm{d}x_1\, \mathrm{d}x_2 = -P; \tag{7}$$

(iii) a condition within the contact region, i.e. for $(x_1, x_2) \in G(P) \cup \partial G(P)$, where ∂G is the boundary of the open region G, namely condition (2) which is rewritten in the form:

$$u_3^+(x_1, x_2, O, P) + u_3^-(x_1, x_2, O, P) = \alpha(P) - f(x_1, x_2), \tag{8}$$

and in addition two more conditions, which depend on considering a contact problem. In the frictionless contact problem for the vector of tangential stresses $\boldsymbol{\tau}(x_1, x_2, P) \equiv (\sigma_{31}(x_1, x_2, O, P), \sigma_{32}(x_1, x_2, O, P))$ we have

$$\tau_\beta(x_1, x_2, P) = 0, \quad \beta = 1, 2. \tag{9}$$

In the adhesive contact problem there is no relative slip between the bodies within the contact region, the values of $v_1 \equiv u_1^+ - u_1^-$ and $v_2 \equiv u_2^+ - u_2^-$ within this region cannot change with augmentation of the force. This is expressed by

$$\frac{\partial}{\partial P} v_\beta(x_1, x_2, O, P) = 0, \quad \mathrm{d}P > 0. \tag{10}$$

In the frictional contact problem it is assumed that the contact region consists of the following parts: in the inner part G_1 the interfacial friction must be sufficient to prevent any slip taking place between the bodies, i.e. eqn (10) holds, and in the outer part $G\backslash G_1$ the friction must satisfy the Coulomb frictional law. These conditions are written as:

$$\frac{\partial}{\partial P} v_\beta(x_1, x_2, O, P) = 0, \quad (x_1, x_2) \in G_1,$$

$$\tau(x_1, x_2, P) = -\theta\sigma_{33}(x_1, x_2, O, P)\left[\frac{\mathbf{v}(x_1, x_2, O, P)}{|\mathbf{v}(x_1, x_2, O, P)|}\right],$$

$$(x_1, x_2) \in G\backslash G_1, \tag{11}$$

where θ is the coefficient of friction (Spektor, 1981).

3. SELF-SIMILAR CONTACT PROBLEMS

Now we highlight the general properties of contact problems. Here we shall use the self-similarity technique. Similarity in problems of mathematical physics (in the case when only scaling is used) is directly connected with the concept of the quasi-homogeneous function [see e.g. Borodich (1988a)] and it is defined below.

Definition (Arnold *et al.*, 1982). The function g with arguments $x_1, \ldots, x_n$ is called a quasi-homogeneous function of degree d with weights $\beta_1, \ldots, \beta_n$ if for any $\lambda > 0$ we have

$$g(\lambda^{\beta_1} x_1, \ldots, \lambda^{\beta_n} x_n) = \lambda^d g(x_1, \ldots, x_n). \tag{12}$$

Evidently, the self-similarity concept follows from the quasi-homogeneous concept. Indeed, let any variable x_i be separated, i.e. it plays the role of a parameter. Let us assume that $i = n$. If we set $\lambda = x_n^{-1/\beta_n}$ then from (12) we have

$$g(x_1, \ldots, x_n) = x_n^{d/\beta_n} g(x_n^{-\beta_1/\beta_n} x_1, \ldots, x_n^{-\beta_{n-1}/\beta_n} x_{n-1}, 1), \tag{13}$$

i.e. we have reduced the number of variables. The functions such as the function in (13) are called the self-similar functions.

If we assume all the functions in the contact problems (3)–(11) to be quasi-homogeneous of different degrees and different weights, then we find that the following conditions are fulfilled:

(i) the function of distance $f(x_1, x_2)$ between two contacting bodies is determined by an arbitrary positive and homogeneous function of degree d, where $d \geqslant 1$, of plane Cartesian coordinates, i.e. the following conditions are satisfied:

$$\begin{gathered} f(x_1, x_2) > 0, \quad (x_1, x_2) \in \mathbf{R}^2\backslash\{0\}, \\ f(x_1, x_2) \in C^1(\mathbf{R}^2\backslash\{0\}), \\ f(\lambda x_1, \lambda x_2) = \lambda^d f(x_1, x_2), \quad d \geqslant 1, \quad \forall\lambda > 0; \end{gathered} \tag{14}$$

(ii) the stress potentials U^+ and U^- for each of the bodies are homogeneous functions of degree $\mu + 1$ with respect to the components of the strain tensor ε_{kl}, i.e.

$$U(\lambda\varepsilon_{kl}) = \lambda^{\mu+1}U(\varepsilon_{kl}), \quad \forall\lambda > 0; \tag{15}$$

(iii) the weights of the arguments x_1, x_2, x_3, P are

$$\beta_1 = 1, \quad \beta_2 = 1, \quad \beta_3 = 1, \quad \beta_4 = a, \quad a \equiv 2+\mu(d-1). \tag{16}$$

We can now formulate the following theorem of self-similarity:

Theorem 1. Let the distance between contacting anisotropic elastic bodies be determined by the function f satisfying (14), and the stress potential u satisfying (15).

Let the functions $\mathbf{u}(\mathbf{x}, P_1)$, $\sigma_{ij}(x, P_1)$, *region* $G(P_1)$ *and quantity* $\alpha(P_1)$ *give the solution of the contact problems (3)–(11) for these bodies.*

In addition, assume $G(P_1)$ *and* $G_1(P_1)$ *are star-shaped regions in* $\mathbf{R}^2$, *then the solution of this problem for any positive force P will be given by:*

$$\mathbf{u}(\mathbf{x}, P) = k^{-d}\mathbf{u}(k\mathbf{x}, P_1), \tag{17}$$

$$\varepsilon_{ij}(\mathbf{x}, P) = k^{1-d}\varepsilon_{ij}(k\mathbf{x}, P_1), \tag{18}$$

$$\sigma_{ij}(\mathbf{x}, P) = k^{-\mu(d-1)}\sigma_{ij}(k\mathbf{x}, P_1), \tag{19}$$

$$\alpha(P) = k^{-d}\alpha(P_1), \tag{20}$$

where

$$k = (P_1/P)^{1/[2+\mu(d-1)]}, \quad \text{i.e. } P_1 = k^a P, \tag{21}$$

and both the contact regions G and G_1 *are changed by the homoteteous transformations:*

$$[(x_1, x_2) \in G(P)] \Leftrightarrow [(kx_1, kx_2) \in G(P_1)],$$
$$[(x_1, x_2) \in G_1(P)] \Leftrightarrow [(kx_1, kx_2) \in G_1(P_1)]. \tag{22}$$

Proof. From (19) one obtains

$$\frac{\partial\sigma_{ij}(\mathbf{x}, P)}{\partial x_j} = k^{-\mu(d-1)}\frac{\partial\sigma_{ij}(\boldsymbol{\xi}, P_1)}{\partial\xi_i}\frac{\partial\xi_i}{\partial x_j},$$

$$\xi_j \equiv kx_j, \quad \partial\xi_j/\partial x_i = k\delta_{ij}, \tag{23}$$

where δ_{ij} is the Kroneker delta.

However, according to the assumption of the theorem, the $\sigma_{ij}(\boldsymbol{\xi}, P_1)$ satisfy (3). Therefore, (23) shows that the $\sigma_{ij}(\mathbf{x}, P)$ satisfy (3) too.

Then using (19) and the assumption that $\sigma_{ij}(\xi, P_1)$ satisfy (4), it can be shown that

$$\sigma_{ij}(\mathbf{x}, P) = k^{-\mu(d-1)}\sigma_{ij}(\boldsymbol{\xi}, P_1) = k^{-\mu(d-1)}\frac{\partial U[\varepsilon_{kl}(\boldsymbol{\xi}, P_1)]}{\partial\varepsilon_{ij}(\boldsymbol{\xi}, P_1)}. \tag{24}$$

On the other hand, using (18) it can be shown that

$$\frac{\partial U[\varepsilon_{kl}(\boldsymbol{\xi}, P_1)]}{\partial\varepsilon_{ij}(\boldsymbol{\xi}, P_1)} = \frac{\partial U[k^{(d-1)}\varepsilon_{kl}(\mathbf{x}, P)]}{\partial\varepsilon_{mn}(\mathbf{x}, P)}\frac{\partial\varepsilon_{mn}(\mathbf{x}, P)}{\partial\varepsilon_{ij}(\boldsymbol{\xi}, P_1)}. \tag{25}$$

By substituting (25) and (15) in (24) one obtains

$$\sigma_{ij}(\mathbf{x},P) = k^{-\mu(d-1)}k^{(\mu+1)(d-1)}\frac{\partial U[\varepsilon_{kl}(\mathbf{x},P)]}{\partial \varepsilon_{ij}(\mathbf{x},P)}k^{(1-d)} = \frac{\partial U[\varepsilon_{kl}(\mathbf{x},P)]}{\partial \varepsilon_{ij}(\mathbf{x},P)}. \tag{26}$$

Therefore, (26) shows that $\sigma_{ij}(\mathbf{x}, P)$ and $\varepsilon_{ij}(\mathbf{x}, P)$ satisfy (4).

The definition of the functions $u_i(\mathbf{x}, P)$ implies that they vanish at infinity.

The validity of the conditions (6)–(8) is proved in the same way:

$$\sigma_{3j}(x_1,x_2,O,P)|_{(x_1,x_2)\in\mathbf{R}^2\backslash G(P)} = k^{-\mu(d-1)}\sigma_{3j}(kx_1,kx_2,O,\ k^aP)|_{(x_1,x_2)\in\mathbf{R}^2\backslash G(P)}$$
$$= k^{\mu(1-d)}\sigma_{3j}(\xi_1,\xi_2,O,P_1)|_{(\xi_1,\xi_2)\in\mathbf{R}^2\backslash G(P_1)} = 0.$$

$$\iint\limits_{G(P)}\sigma_{33}(x_1,x_2,O,P)\,\mathrm{d}x_1\,\mathrm{d}x_2 = k^{-2}\iint\limits_{G(P)}k^{-\mu(d-1)}\sigma_{33}(kx_1,kx_2,O,k^aP)\,\mathrm{d}(kx_1)\,\mathrm{d}(kx_2)$$
$$= k^{-\mu(d-1)-2}\iint\limits_{G(P_1)}\sigma_{33}(\xi_1,\xi_2,O,P_1)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 = -k^{-[\mu(d-1)+2]}P_1 = -P.$$

Let $(x_1, x_2)\in G(P)\cup\partial G(P)$, then one has

$$[u_3^+(\mathbf{x},P)+u_3^-(\mathbf{x},P)]|_{x_3=0} = k^{-d}[u_3^+(\boldsymbol{\xi},P_1)-u_3^-(\boldsymbol{\xi},P_1)]|_{\xi_3=0}$$
$$= k^{-d}[\alpha(P_1)-f(kx_1,kx_2)] = \alpha(P)-f(x_1,x_2).$$

The conditions (9)–(11) should be considered separately.

From (9) we have

$$\tau_\beta(x_1,x_2,P) = k^{-\mu(d-1)}\sigma_{3\beta}(\xi_1,\xi_2,O,P_1) = 0,\quad (\xi_1,\xi_2)\in\mathbf{R}^2\backslash G(P_1)$$

and the theorem in the case of the frictionless contact problem has been proved.

In the case of adhesive contact the inner region G is equal to the whole contact region G. Note that (17)–(21) show that the functions $u_i(\mathbf{x}, P)$ are quasi-homogeneous functions of degree d with weights $(1, 1, 1, a)$. Then for every positive quantity λ one has

$$\frac{\partial u_i(\mathbf{x},P)}{\partial P} = \lambda^{-d}\frac{\partial u_i(\lambda\mathbf{x},\lambda^aP)}{\partial P} = \lambda^{-d}\frac{\partial u_i(\lambda\mathbf{x},\lambda^aP)}{\partial(\lambda^aP)}\lambda^a = \lambda^{a-d}\frac{\partial u_i(\lambda\mathbf{x},\lambda^aP)}{\partial(\lambda^aP)}.$$

Next, let $\lambda = k = (P_1/P)^{1/a}$ then

$$\frac{\partial u_i(\mathbf{x},P)}{\partial P} = k^{a-d}\frac{\partial u_i(k\mathbf{x},P_1)}{\partial P_1}.$$

This and (22) ensure that on the surface $x_3 = 0$ one has

$$\left.\frac{\partial u_\beta(x_1,x_2,O,P)}{\partial P}\right|_{(x_1,x_2)\in G_1(P)} = k^{a-d}\left.\frac{\partial u_\beta(\xi_1,\xi_2,O,P_1)}{\partial P_1}\right|_{(\xi_1,\xi_2)\in G_1(P_1)}. \tag{27}$$

Finally, (27) together with the assumption that condition (10) is satisfied for P_1, implies

$$\frac{\partial v_\beta(x_1,x_2,O,P)}{\partial P} = 0,\quad (x_1,x_2)\in G_1(P).$$

In the case of the frictional problem the contact region consists of the inner adhesive region and the outer annulus of slip. For the adhesive region $G_1(P)$ the boundary condition is verified in the same way as in the adhesive problem and we should verify the validity of boundary condition (11) in the outer region $R^2\backslash G_1$ only. From (17), (19) and (22) one obtains

$$-\theta\sigma_{33}(x_1,x_2,O,P)\left[\frac{\mathbf{v}(x_1,x_2,O,P)}{|\mathbf{v}(x_1,x_2,O,P)|}\right] = -\theta k^{-\mu(d-1)}\sigma_{33}(\xi_1,\xi_2,O,P_1)\left[\frac{\mathbf{v}(\xi_1,\xi_2,O,P_1)}{|\mathbf{v}(\xi_1,\xi_2,O,P_1)|}\right] \tag{28}$$

and

$$\boldsymbol{\tau}(x_1,x_2,P) = k^{-\mu(d-1)}\boldsymbol{\tau}(\xi_1,\xi_2,P_1).$$

However, according to the assumption of the theorem the vector of tangential stresses $\boldsymbol{\tau}(\xi_1, \xi_2, P_1)$ satisfies (11). Therefore, (28) shows that $\tau_\beta(x_1, x_2, P)$ satisfies (11) too.

The proof of Theorem 1 is now complete.

The theorem has pointed out the classes of self-similar contact problems which include all previously considered self-similar contact problems of three-dimensional infinitesimal elasticity as particular cases [see e.g. Spence (1968, 1975), Galanov (1981), Borodich (1983, 1988b, 1989, 1990), Hill *et al.* (1989), Storåkers (1989) and Hill and Storåkers (1990)].

The theorem provides the following qualitative corollaries which hold under the assumptions that have been made above:

(i) The size l of the contact region varies in proportion to the load raised to the power $1/a$;

(ii) The approach of the bodies is proportional to the load raised to the power d/a, namely:

$$l(P) = [l(P_1)P_1^{-1/[2+\mu(d-1)]}]\,P^{1/[2+\mu(d-1)]}, \tag{29}$$

$$\alpha(P) = [\alpha(P_1)P_1^{-d/[2+\mu(d-1)]}]\,P^{d/[2+\mu(d-1)]}. \tag{30}$$

Indeed, the definition of the region $G(P)$ by (22) implies that the size of the contact area varies proportionally to k^{-1} and this yields, after substituting from (21), the first assertion. The second assertion follows from (20).

Note that from the solutions of the problems of contact between linear elastic bodies in which $d = 2$ (Hertz, 1881; Willis, 1966) and from such solutions in which $d = 2n$ and $d = s/n$, where s and n are natural [see e.g. Shtaerman (1939) and Galin (1953)] formulae analogous to (29) and (30).

We now give the functional forms of the relative tangential displacements $v_\beta(x_1, x_2, O, P)$ within the contact region in the contact problems considered.

From the conditions of adhesive contact one has

$$v_\beta(x_1,x_2,O,P) = v_\beta^0(x_1,x_2), \quad (x_1,x_2)\in G(P), \tag{31}$$

where v_β^0 are certain functions of x_1 and x_2.

On the other hand $v_\beta(x_1, x_2, O, P)$ are the quasi-homogeneous functions of degree d with weights $(1, 1, 1, a)$. Then for every positive quantity λ one has

$$v_\beta(\lambda x_1,\lambda x_2,O,\lambda^a P) = \lambda^d v_\beta(x_1,x_2,O,P). \tag{32}$$

From (31) and (32) follows

$$v_\beta(x_1,x_2,O,P) = \lambda^{-d}v_\beta^0(\lambda x_1,\lambda x_2), \quad (x_1,x_2)\in G(P). \tag{33}$$

In the polar coordinate system r, $\varphi(x_1 = r\cos\varphi,\ x_2 = r\sin\varphi)$ by substituting $\lambda = r^{-1}$ in (33) one has for $(x_1, x_2)\in G_1(P)$:

$$v_\beta(x_1, x_2, O, P) = r^d v_\beta^0(\cos\varphi, \sin\varphi) = r^d w_\beta(\varphi), \tag{34}$$

where w_β are certain functions of the angle φ.

Note, that in the axisymmetric problem of adhesive contact of isotropic bodies the corollary (34) was obtained by Spence (1968).

Above we assumed the resultants of compressive forces to lie always on the x_3-axes and therefore the moments of contact stresses about the x_1- and x_2-axes are both zero. But this assumption may be weakened in the following way:

Let moments $M_1(P_1)$ and $M_2(P_1)$ both be zero for the compressive force P_1, i.e.

$$M_\beta(P_1) = (-1)^{\beta+1} \iint_{G(P_1)} x_\beta \sigma_{33}(x_1, x_2, O, P_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0. \tag{35}$$

Then, in the self-similar problems considered above, $M_1(P)$ and $M_2(P)$ are both zero for any positive compressive force P. Indeed, by substituting (19) and (22) into (35), one has

$$(-1)^{\beta+1} M_\beta(P) = \iint_{G(P)} x_\beta \sigma_{33}(x_1, x_2, O, P)\,\mathrm{d}x_1\,\mathrm{d}x_2$$

$$= k^{-3} \iint_{G(P)} (kx_\beta) k^{-\mu(d-1)} \sigma_{33}(kx_1, kx_2, O, k^a P) d(kx_1) d(kx_2)$$

$$= k^{-\mu(d-1)-3} \iint_{G(P_1)} \xi_\beta \sigma_{33}(\xi_1, \xi_2, O, P_1)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 = (-1)^{\beta+1} k^{-\mu(d-1)-3} M_\beta(P_1) = 0.$$

Finally, note that although the actual contact boundary-value problem is non-steady it can be made steady in terms of reduced variables when the problem is self-similar in the way shown above [see eqn (13)]. Galanov (1981) showed in an example for an isotropic medium that it is very convenient for numerical calculations of the stress and strain fields to rewrite the contact problem in terms of reduced variables.

4. FRICTIONAL COLLISION OF ANISOTROPIC BODIES

In the same way as in the frictionless contact problems (Hertz, 1881; Willis, 1966) one can extend Hertz's theory of impact to the frictional impact of two anisotropic nonlinear bodies.

Let the bodies have masses m^+ and m^-. Hertz's approximation is that, during the impact, the distribution of stresses for any α is the same as that obtained from the solution of the corresponding static contact problem (Willis, 1966). Then, considering the motion of the mass center of each body, we obtain:

$$\frac{m^+ m^-}{m^+ + m^-} \ddot{\alpha}(P) = -P. \tag{36}$$

From (30) one has

$$P = [P_1 \alpha^{-[2+\mu(d-1)]/d}(P_1)]\alpha^{[2+\mu(d-1)]/d}(P). \tag{37}$$

By substituting (37) into (36), we have

$$\ddot{\alpha} = -k_1 k_2 \alpha^{[2+\mu(d-1)]/d}; \quad k_1 = \frac{m^+ + m^-}{m^+ m^-}, \quad k_2 = P_1 \alpha^{-[2+\mu(d-1)]/d}(P_1). \tag{38}$$

Multiplying both sides by $\dot{\alpha}$ and integrating (provided $\dot{\alpha} = V$ for $t = 0$, where V is the relative speed of the bodies just before their collision) we obtain:

$$\dot{\alpha}^2 - V^2 = -\frac{2d}{2+d+\mu(d-1)} k_1 k_2 \alpha^{[2+d+\mu(d-1)]/d}. \tag{39}$$

At the moment of maximum compression the relative speed $\dot{\alpha}$ vanishes. Then the maximal approach of the bodies α_* is given by:

$$\alpha_* = \left(\frac{2+d+\mu(d-1)}{2d} \frac{1}{k_1 k_2}\right)^{d/[2+d+\mu(d-1)]} V^{2d/[2+d+\mu(d-1)]}. \tag{40}$$

This value is achieved at the moment t_* which is calculated from (39):

$$t_* = \int_0^{\alpha_*} \frac{d\alpha}{\left[V^2 - \frac{2d}{2+d+\mu(d-1)} k_1 k_2 \alpha^{[2+d+\mu(d-1)]/d}\right]^{1/2}}$$

$$= \frac{d}{2+d+\mu(d-1)} \sqrt{\pi} \frac{\alpha_*}{V} \frac{\Gamma\left[\frac{d}{2+d+\mu(d-1)}\right]}{\Gamma\left[\frac{2+3d+\mu(d-1)}{4+2d+2\mu(d-1)}\right]}. \tag{41}$$

From (37) we can show that the maximum compressive force between bodies P_{max} is

$$P_{max} = k_2 \alpha_*^{[2+\mu(d-1)]/d}, \quad P_{max}(V) = P_{max}(V_1)\left(\frac{V}{V_1}\right)^{(2[2+\mu(d-1)])/(2+d+\mu(d-1))}. \tag{42}$$

As an example, we consider the collision between elasto-plastic bodies. Such consideration is possible because the deformation theory of plasticity for an active process conforms to the physically nonlinear theory of elasticity. Experimental points of the $P_{max} \sim V$ curve, which are obtained for a collision between flat and spherical soft steel surfaces ($d = 2$) with radius 14.5 mm, (Bagreev, 1963), are presented in Fig. 1.

The experimental stress–strain curve for soft steel under compression can be described by the relationship $\sigma \sim \varepsilon^{0.6}$ in the interval of $0.01 < \varepsilon < 0.04$. If we take $V_1 = 10$ cm s^{-1}

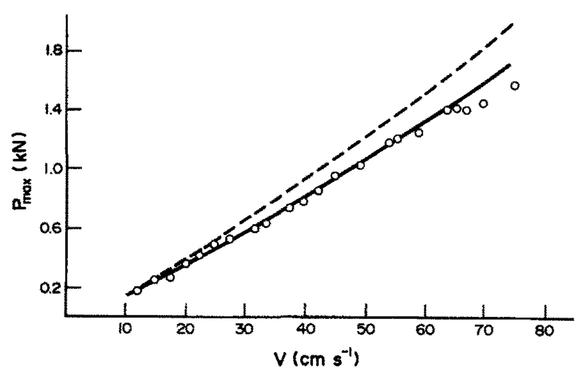

Fig. 1. Relation between maximal compressive force P_{max} and relative speed of collision V obtained from (42) where $P_{max}(V_1) = 175$ N, $V_1 = 10$ cm s^{-1}. (———) $d = 2$, $\mu = 0.6$. (– – –) $d = 2$, $\mu = 1$ (Hertz solution). ○—experimental data for soft steel specimens from Bagreev (1963).

and $P_{max} = 175$ N, we can draw the curves corresponding to computation from eqn (42) and to the Hertz solution when $\mu = 0.6$ and $\mu = 1$, respectively. Good agreement is obtained between the first curve and the experimental points, as Fig. 1 shows.

Note that eqn (42) is independent of the choice of boundary conditions (9)–(11).

5. A TWO-PARAMETER TRANSFORMATION OF SIMILARITY

Let us now consider a more general transformation of the initial contact problem (for simplicity we shall consider a rigid punch pressing in a nonlinear half-space). In this transformation one punch is replaced by another.

Let the function of the shape $f_1(x_1, x_2)$ of the first punch be transformed by tension λ_1 times along the x_1 and x_2 axes and λ_2 times along the x_3 axis (see Fig. 2), i.e. the function of the shape $f(x_1, x_2)$ of the second punch is given by

$$f(x_1, x_2) = \lambda_2 f_1(\lambda_1^{-1} x_1, \lambda_1^{-1} x_2). \tag{43}$$

Then we are able to give a formulation of the following theorem of similarity.

Theorem 2. Let the shape of a punch be determined by any positive function f_1.

Let the punch be pressed in an anisotropic nonlinear elastic half-space with stress potential U satisfying (15).

Let the functions $u_i^*(\mathbf{x}, P_1)$, $\sigma_{ij}^*(\mathbf{x}, P_1)$, *the quantity* $\alpha^*(P_1)$ *and the regions* $G^*(P_1)$ *and* $G_1^*(P_1)$ *give the solution of the contact problems (3)–(11) for this punch and a pressing force* P_1.

In addition, assume $G^*(P_1)$ *and* $G_1^*(P_1)$ *are star-shaped regions in* $\mathbf{R}^2$.

Then, the solution of this problem for the other punch, whose shape is determined by function f satisfying (43), pressed in the half-space by the force

$$P = \lambda_1^{(2-\mu)} \lambda_2^{\mu} P_1 \tag{44}$$

will be given by $\mathbf{u}(\mathbf{x}, P)$, $\sigma_{ij}(\mathbf{x}, P)$, $\alpha(P)$, *namely*

$$u_i(\mathbf{x}, P) = \lambda_2 u_i^*(\lambda_1^{-1}\mathbf{x}, P_1), \tag{45}$$

$$\sigma_{ij}(\mathbf{x}, P) = \left(\frac{\lambda_2}{\lambda_1}\right)^{\mu} \sigma_{ij}^*(\lambda_1^{-1}\mathbf{x}, P_1), \tag{46}$$

$$\alpha(P) = \lambda_2 \alpha^*(P_1) \tag{47}$$

and both contact regions $G(P)$ *and* $G_1(P)$ *are altered by homoteteous transformations*

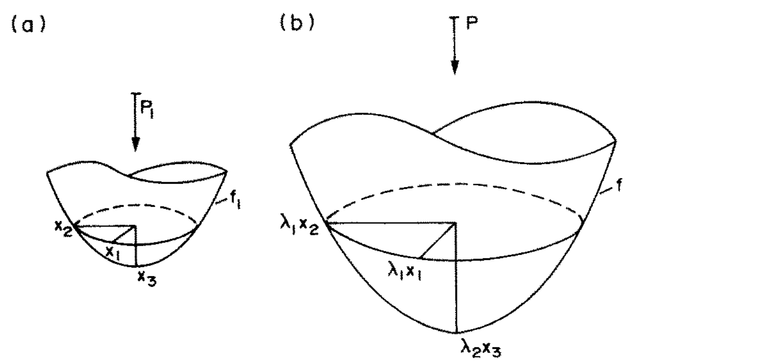

Fig. 2. Transformation of similarity with parameters λ_1 and λ_2: (a) Initial punch compressed by a load P_1; (b) transformed by (43) punch compressed by a load $P = \lambda_1^{(2-\mu)}\lambda_2^{\mu}P_1$.

$$[(x_1,x_2)\in G(P)] \Leftrightarrow [(\lambda_1^{-1}x_1,\lambda_1^{-1}x_2)\in G^*(P_1)],$$
$$[(x_1,x_2)\in G_1(P)] \Leftrightarrow [(\lambda_1^{-1}x_1,\lambda_1^{-1}x_2)\in G_1^*(P_1)]. \tag{48}$$

The proof of Theorem 2 is demonstrated in the same way as the proof of Theorem 1, i.e. by the direct verification of the validity of all conditions (3)–(11) for the collection of the functions given by (43)–(48).

Note that if the shape function of punch is determined by an arbitrary positive and homogeneous function of degree d, i.e. it is satisfied by conditions (14), then Theorem 1 follows from Theorem 2. Indeed, if we take $\lambda_1 = k^{-1}$, and $\lambda_2 = k^{-d}$, then (43)–(48) transform in (17)–(22).

6. A THEORETICAL STUDY OF HARDNESS TESTS

Hardness tests (Brinell, 1901; Meyer, 1908, and others) have long been the preferred method of assaying the mechanical properties of metals during forming operations [see references in Hill *et al.* (1989) and Borodich (1989)].

In practice, it is frequently necessary to investigate the character of the interaction between spherical punches and a physically nonlinear foundation. A new series of experiments must be performed, however, when the radius of the sphere is changed.

Meyer's empirical hardness law shows

$$P \sim l^{\kappa}, \tag{49}$$

where κ is some material constant. In the case of spherical punches this constant lies in the interval $2 < \kappa < 3$. The semi-empirical formula of Bowden and Tabor (1964) shows that

$$P \sim l^{2+m}R^{-m}, \tag{50}$$

where m is another material constant and R is the radius of the punch.

Here, we obtain the exact formulae similar to the formulae (49), (50).

Let the shape of a punch be determined by the homogeneous function f_1 of degree d. We have from a test for the punch under load P_1 the quantities $l(1, P_1)$ and $\alpha(1, P_1)$.

Let us assume another punch, which is bigger than the first by a factor of c, i.e. its shape is determined by the function f such that $f(x_1, x_2) = cf_1(x_1, x_2)$.

Suppose that the material is nonlinear elastic with stress potential U satisfying (15).

Then the depth of indentation $\alpha(c, P)$ and size of contact region $l(c, P)$ in the test for the second punch under another load P will be given by

$$l(c,P) = c^{-\mu/[2+\mu(d-1)]}(P/P_1)^{1/[2+\mu(d-1)]}l(1,P_1), \tag{51}$$

$$\alpha(c,P) = c^{(2-\mu)/[2+\mu(d-1)]}(P/P_1)^{d/[2+\mu(d-1)]}\alpha(1,P_1). \tag{52}$$

Indeed, it follows from Theorem 2 that in the case under consideration $c = \lambda_1^{-d}\lambda_2$ and $P/P_1 = \lambda_1^{2-\mu}\lambda_2^{\mu}$, i.e.

$$\lambda_1 = c^{-[\mu/[2+\mu(d-1)]]}(P/P_1)^{1/[2+\mu(d-1)]}, \tag{53}$$

$$\lambda_2 = c^{1-[\mu d/[2+\mu(d-1)]]}(P/P_1)^{d/[2+\mu(d-1)]}. \tag{54}$$

Then substitution from (53) and (54) into (47) and (48) results in (51) and (52).

Note, that the formulae (51) and (52) are exact under the assumptions made above.

Evidently, the formulae (49) and (50) are the particular cases of the formulae (51) and (52). Indeed, let the punches be spherical with radii R_1 and R, respectively. Then $d = 2$ and c is equal to R_1/R. Using (51) and (52), we obtain

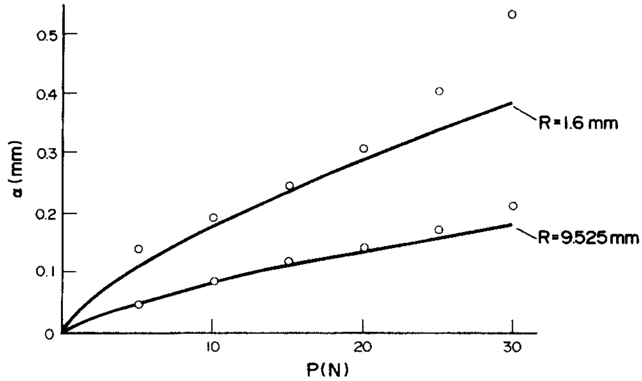

Fig. 3. Relation between the depth α of indentation into anisotropic balsa wood specimen and load P obtained from (56) where $R_1 = 9.525$ mm, $P_1 = 10$ N. ○—experimental data from Bowden and Tabor (1964).

$$l(R, P) = (R/R_1)^{\mu/(2+\mu)}(P/P_1)^{1/(2+\mu)}l(R_1, P_1), \tag{55}$$

$$\alpha(R, P) = (R_1/R)^{(2-\mu)/(2+\mu)}(P/P_1)^{2/(2+\mu)}\alpha(R_1, P_1). \tag{56}$$

The last formulae were obtained for the frictionless case by Borodich (1989). Comparison between (55) and (49), (50) leads to $\kappa = 2+\mu$ and $m = \mu$.

As an example, we examine the test of the penetration of spherical punches into the surface of an anisotropic wooden specimen. The results of this test are presented by Bowden and Tabor (1964). Owing to the material's anisotropy (balsa wood, 67% saturation), the contact region G is not a circle but resembles an ellipse. Bowden and Tabor (1964) showed that $l \sim P^{1/2.8}$. It follows from the comparison of this relationship and eqn (55) that $\mu = 0.8$. Taking only one value of the depth of penetration from the test: $\alpha(R_1, P_1) = 81.6 \cdot 10^{-6}$ m where $P_1 = 10$ N and $R_1 = 9.525$ mm, we can now determine the other values of $\alpha(R, P)$ using eqn (56). Experimental and theoretical $\alpha \sim P$ curves plotted by the method described are shown in Fig. 3.

Note that the formulae (51), (52), (55) and (56) are valid for any of the boundary conditions (9)–(11).

7. CONCLUSION

The actual contact boundary-value problem is non-steady but can be made steady in terms of reduced variables when the problem is self-similar (Galanov, 1981; Borodich, 1983, 1988b, 1989, 1990; Hill *et al.*, 1989; Storåkers, 1989; Hill and Storåkers, 1990). The main condition of self-similarity is the condition that the distance between bodies is a homogeneous function. Such functions are numerous. Quadratic forms and the fourth degree forms are particularly special examples of such functions. And the classical Hertz problem of contact, when it is assumed that the function of the distance is a quadratic form, lies within the class pointed out.

We have shown that if the contact problem is self-similar then its properties are independent of the choice of boundary conditions and are validity for both linear and nonlinear, isotropic and anisotropic media. So, one can say that the self-similarity law of change of the solutions is the general property of the considered problems.

REFERENCES

Arnold, V. I., Varchenko, A. N. and Gusein-Zade, S. M. (1982). Singularities of differentiable maps. Nauka, Moscow. (English translation by Birkhäuser, Boston, Basel, 1985.)

Bagreev, V. V. (1963). Some characteristic features of elasto–plastic impact. *Proc. Moscow Institute of Railroad Transport*. No. 174, 113–117.

Borodich, F. M. (1983). Similarity in the problem of contact between elastic bodies. *J. Appl. Math. Mech. (PMM)* **47**(3), 440–442. (English translation 1984.)

Borodich, F. M. (1988a). Three-dimensional self-similar problems involving the penetration of blunt body in an ideal compressible fluid (supersonic case). *J. Appl. Mech. and Tech. Phys.* **29**(3), 415–418.
Borodich, F. M. (1988b). Using of the theory of similarity in the nonlinear problem of contact between indentor and anisotropic metallic foundation. In *Abstracts of Reports of All-Union Conference "Metal"-programme's fulfillers* (Edited by A. A. Bogatov *et al.*) pp. 195–196 (in Russian). Krasnoyarskiy Polytechnical Institute Press, Abakan.
Borodich, F. M. (1989). Hertz contact problems for an anisotropic physically-nonlinear elastic medium. *Strength of Materials* No. 12, 1668–1676. (English translation 1990.)
Borodich, F. M. (1990). Three-dimensional problems of contact of blunt bodies with continuous media. Dr. Phys.-Math. Sc. Dissertation, Moscow University.
Bowden, F. P. and Tabor, D. (1964). *Friction and Lubrication of Solids.* Oxford University Press, Oxford.
Bryant, M. D. and Keer, L. M. (1982). Rough contact between elastically and geometrically identical curved bodies. *ASME J. Appl. Mech.* **49,** 345–352.
Galanov, B. A. (1981). Approximate solution of some contact problems with an unknown contact area under conditions of power law of material hardening. *Dokl. AN Ukrain. SSR*, **Ser. A**, No. 6, 36–41 (in Russian).
Galin, L. A. (1953). *Contact Problems in the Theory of Elasticity.* Gostechisdat, Moscow (English translation edited by I. N. Sneddon, North Carolina State College. Dep. Math. Report 1961).
Gladwell, G. M. L. (1980). *Contact Problems in the Classical Theory of Elasticity.* Sijthoff and Noordhoff, Alphen aan den Rijn.
Goodman, L. E. (1962). Contact stress analysis of rough spheres. *ASME J. Appl. Mech.* **54,** 515–522.
Hertz, H. (1881). Über die Berürung fester elastischer Körper. *J. reine angewandte Math.* **92,** 156–171.
Hill, R. and Storåkers, B. (1990). A concise treatment of axisymmetric indentation in elasticity. In *Elasticity: Mathematical Methods and Applications* (Edited by G. Eason and R. W. Ogden), pp. 199–209. Chichester, Ellis Horwood.
Hill, R., Storåkers, B. and Zdunek, A. B. (1989). A theoretical study of the Brinell hardness test. *Proc. R. Soc. Lond. A* **423,** 301–330.
Hills, D. A. and Sackfield, A. (1987). The stress field induced by normal contact between dissimilar spheres. *ASME J. Appl. Mech.* **54,** 8–14.
Johnson, K. L. (1985). *Contact Mechanics.* Cambridge University Press, Cambridge.
Kalker, J. J. (1985). On the contact problem in elastostatics. *Cours. and Lect. CISM. Int. Cent. Mech. Sci.* No. 288, 81–118.
Keer, L. M. (1967). Mixed boundary-value problems for an elastic half-space. *Proc. Camb. Phil. Soc.* **63,** 1379–1386.
Kil'chevskii, N. A. (1976). *Dynamic Contact Compression of Solids. Impact.* Naukova Dumka, Kiev (in Russian).
Lekhnitskii, S. G. (1950). *Theory of Elasticity of an Anisotropic Elastic Body.* Gostekhizdat, Moscow (English translation Holden-Day, San Francisco, 1963).
Mossakovskii, V. I. (1954). The fundamental mixed problem of the theory of elasticity for a half-space with a circular line separating the boundary conditions. *J. Appl. Math. Mech. (PMM)* **18**(2), 187–196 (in Russian).
Mossakovskii, V. I. (1963). Compression of the elastic bodies under conditions of adhesion (axisymmetric case). *J. Appl. Math. Mech. (PMM)* **27**(3), 418–427.
Shtaerman, I. Ya. (1939). To Hertz's theory of local deformations in compressed elastic bodies. *Dokl. AN SSSR* **25**(5), 360–362 (in Russian).
Spektor, A. A. (1981). Some three-dimensional static contact problems of elasticity with slippage and adhesion. *Mech. Solids* **16**(3), 10–21.
Spence, D. A. (1968). Self similar solutions to adhesive contact problems with incremental loading. *Proc. R. Soc.* **A305,** 55–80.
Spence, D. A. (1975). The Hertz contact problem with finite friction. *J. Elasticity* **5,** 297–319.
Storåkers, B. (1989). Some self-similarity features of Hertzian contact of nonlinear solids. Royal Institute of Technology, Department of Solid Mechanics. Internal Report.
Turner, J. R. (1980). Contact on a transversely isotropic half-space, or between two transversely isotropic bodies. *Int. J. Solids Structures* **16,** 409–419.
Vermeulen, P. J. and Johnson, K. L. (1964). Contact of nonspherical elastic bodies transmitting tengential forces. *ASME J. Appl. Mech.* **31,** 338–340.
Willis, J. R. (1966). Hertzian contact of anisotropic bodies. *J. Mech. Phys. Solids* **14,** 163–176.

APPENDIX. COLLISION OF TWO LINEAR TRANSVERSELY ISOTROPIC BODIES

In Hertz's approximation the shapes of the bodies are described by quadratic forms and thus f is a homogeneous function of degree $d = 2$. Then there is the exact transfer of Hertz's qualitative conclusions on the character of the duration of the impact and the maximal approach:

$$\alpha_* = \left(\frac{5}{4}\frac{1}{k_1 k_2}\right)^{2/5} V^{4/5}, \quad T = \left[\left(\tfrac{4}{5}\right)^{3/5}\sqrt{\pi}\frac{1}{(k_1 k_2)^{2/5}}\frac{\Gamma(\frac{2}{5})}{\Gamma(\frac{9}{10})}\right] V^{-1/5}. \tag{A1}$$

Indeed, these conclusions follow from (40)–(41) with $d = 2$.

Consider the adhesive collision of elastic spheres with radii R^+ and R^-. Then in terms of polar coordinates one can write

$$f^+ + f^- = \frac{1}{2R} r^2, \tag{A2}$$

where

$$R = \frac{R^+ R^-}{R^+ + R^-}.$$

Let the materials of the spheres be transversely isotropic for which the preferred directions are normal to the Ox_1x_2-surface. Then there are only five independent nonzero components of the compliance tensor. Using the engineering components of the compliance tensor given by Lekhnitskii (1950):

$$E\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{13}\\ \varepsilon_{23}\\ \varepsilon_{12}\end{bmatrix}=\begin{bmatrix}1 & -\nu_H & -\nu_V & & & \\ -\nu_H & 1 & -\nu_V & & & \\ -\nu_V & -\nu_V & \lambda & & & \\ & & & 2(1+\nu) & & \\ & & & & 2(1+\nu) & \\ & & & & & 2(1+\nu_H)\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{13}\\ \sigma_{23}\\ \sigma_{12}\end{bmatrix},$$

where $2\varepsilon_{ij} = u_{i,j}+u_{j,i}$.

Introducing the elastic parameters a, b, c, γ, and δ (Turner, 1980):

$$a=\left[\frac{\lambda-\nu_V^2}{1-\nu_H^2}\right]^{1/2},$$

$$b=\frac{1+\nu-\nu_V(1+\nu_H)}{1-\nu_H^2},$$

$$c=\left(\frac{a+b}{2}\right)^{1/2}\frac{2(1-\nu_H)(1+\nu_H)}{E},$$

$$\gamma=\left(\frac{a+b}{2}\right)^{-1/2}\left[\frac{a(1-\nu_H)-\nu_V}{2(1-\nu_H)}\right],$$

$$\delta=\left[\left(\frac{a+b}{2}\right)^{-1/2}\left(\frac{1+\nu}{1+\nu_H}\right)^{1/2}\left(\frac{1}{1-\nu_H}\right)-1\right],$$

and also the parameters A, C, D, Δ:

$$C=c^++c^-,\qquad A=(a^+c^++a^-c^-)/C,$$
$$D=(\gamma^+c^+-\gamma^-c^-)/C,\quad \Delta=(\delta^+c^++\delta^-c^-)/C.$$

Turner (1980) obtained from Spence's solution the exact solution of the adhesive contact problem for transversely isotropic spheres. From Turner's solution we get for transversely isotropic materials

$$k_2=\tfrac{4}{3}\pi\frac{k_*}{DC\sqrt{RA}}\Phi^{3/2}(k_*), \tag{A3}$$

where

$$k_*\equiv\frac{1}{\pi}\ln\left[\frac{A^{1/2}+D}{A^{1/2}-D}\right],$$

$$\Phi(k_*)=1-k_*^2(\operatorname{cosech}\tfrac{1}{2}\pi k_*)\int_0^{\pi/2}(\cot\xi\,\sinh k_*\xi)\,\mathrm{d}\xi=1-0.6931k_*^2+0.2254k_*^4+\cdots.$$

Finally, (A3) together with (A1) and (A2), gives the exact solutions of Hertz adhesive collision problems for isotropic or transversely isotropic spheres.

For the isotropic materials $\lambda = 1$, $\nu=\nu_H=\nu_V$, $a=b=1$, $c=(1-\nu)/\mu$, $\delta=\nu/(1-\nu)$, where μ are shear moduli of the spheres. Thus, we have

$$A=1,\quad C=\frac{1+\nu^+}{\mu^+}+\frac{1-\nu^-}{\mu^-},\quad D=\frac{\left(\dfrac{1-2\nu^+}{2\mu^+}-\dfrac{1-2\nu^-}{2\mu^-}\right)}{\left(\dfrac{1-\nu^+}{\mu^+}+\dfrac{1-\nu^-}{\mu^-}\right)}$$

and

$$k_2=\tfrac{4}{3}\pi\frac{k_*}{\left(\dfrac{1-2\nu^+}{2\mu^+}-\dfrac{1-2\nu^-}{2\mu^-}\right)\sqrt{R}}.$$

Note, that one could also obtain this value of the constant k_2 from Spence's solution.